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Solvable symmetric black holes in anti-de Sitter spaces

Laurent Claessens[a,*,1], Stéphane Detournay[b,1]

[a] Département de Mathématique, Université Catholique de Louvain, Chemin du cyclotron, 2, 1348 Louvain-La-Neuve, Belgium
[b] Mécanique et Gravitation, Université de Mons-Hainaut, 20 Place du Parc, 7000 Mons, Belgium

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Dedicated to Jeanne

Abstract

Using symmetric space techniques, we show that closed orbits of Iwasawa subgroups of $SO(2, l-1)$ naturally define the singularity of a black hole causal structure in anti-de Sitter spaces in $l \geq 3$ dimensions. In particular, we recover for $l = 3$ the non-rotating massive BTZ black hole. The method presented here is very simple and in principle generalizable to any semi-simple symmetric space.

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1. Introduction

From a geometric point of view, a *black hole* is the data of a time orientable pseudo-Riemannian manifold M together with a subset $\mathscr{S} \subset M$ called a *singularity* in such a way that the whole manifold is divided into two parts: the *interior* and the *exterior* of the black hole. A point is said to be *interior* if all future geodesics through the point have a non-empty intersection with the singularity. A point is *exterior* if it is not interior. An important subset of the space is the *horizon*: the boundary between these two parts.

Most black hole singularities usually emerge from metric considerations: a singularity corresponds to the set of points where a metric invariant diverges. This is however not always the case, as shown for example in the BTZ black holes [5,4]. The latter are obtained as quotients of three-dimensional anti-de Sitter space (AdS_3), identified with the (universal covering of the) group manifold $SL_2(\mathbb{R})$, under the action of particular subgroups of its isometry group. Regions where the orbits of the identification subgroup are time-like are excluded from the original space

* Corresponding author.
E-mail address: claessens@math.ucl.ac.be (L. Claessens).
1 "Chercheur FRIA", Belgium.

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in order to avoid closed time-like curves. The singularity then corresponds to the region where the identifications become light-like, and it can be shown that the resulting space is indeed a black hole. It was further noticed [11, 8–10] that for a particular subclass of BTZ black holes, the non-rotating massive ones, singularity and horizons have a simple group-theoretical interpretation: they correspond to unions of minimal parabolic (solvable) subgroups of $SL_2(\mathbb{R})$, and their translated. It is worth remarking that seeing $SL_2(\mathbb{R})$ as the homogeneous space $SO(2,2)/SO(2,1)$, the singularity can be identified with the closed orbits of minimal parabolic subgroups of the isometry group $SO(2,2) \sim (SL_2(\mathbb{R}) \times SL_2(\mathbb{R}))/\mathbb{Z}_2$. This type of homogeneous space furthermore belongs to the general class of causal symmetric spaces (for a definition and examples, see [12,14,17]).

This observation motivates the following definition:

Definition 1. A *causal solvable symmetric black hole* is a causal symmetric space where the closed orbits of minimal parabolic subgroups of its isometry group define a black hole singularity.

In this situation, the black hole causal structure is thus completely determined by the action of a solvable group. This observation gives prominence to potential embeddings of these spaces in the framework of noncommutative geometry, in defining noncommutative causal black holes (see also [9]) through the existence of universal deformation formulae for solvable group actions which have been obtained in the context of WKB quantization of symplectic symmetric spaces [6,7].

Non-rotating massive BTZ black holes turn out to enter the class of causal symmetric solvable black holes. The purpose of this paper is to generalize known results about the three-dimensional case to anti-de Sitter spaces of arbitrary dimension, in proving the following theorem:

Theorem 2. *For all $l \geq 3$, anti-de Sitter space in l dimensions, seen as the symmetric space $SO(2,l-1)/SO(1,l-1)$, becomes a causal symmetric solvable black hole, as defined above, when closed orbits of a minimal parabolic subgroup of $SO(2,l-1)$ and its Cartan conjugate are said to be singular.*

This paper intends to prove this theorem. It could be interesting to relate this construction to previous ones giving rise to black holes in AdS spaces; see [13,18,3,2,1,15].

2. Symmetric structure

For the sake of notational simplicity, we put $G = SO(2,l-1)$ and $H = SO(1,l-1)$; our space of interest is $\mathrm{AdS}_l = G/H$. The equivalence class of $g \in G$ is denoted by $[g]$ and $\vartheta = [e]$. The anti-de Sitter space can be seen as the surface

$$M \equiv u^2 + t^2 - x^2 - y^2 - x_3^2 - \cdots - x_{l-1}^2 = 1$$

embedded in $\mathbb{R}^{2,l-1}$.

When H is the $SO(1,l-1)$ subgroup of $SO(2,l-1)$ which leaves unchanged the vector $(1,0,\ldots,0) \in \mathbb{R}^{2,l-1}$, the isomorphism is given by

$$[g] \to g \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where the dot means the usual matrix time vector product in $\mathbb{R}^{l+1}$. The Lie algebras of G and H will be denoted by $\mathcal{G}$ and $\mathcal{H}$ respectively. A Cartan involution $\theta : \mathcal{G} \to \mathcal{G}$ gives rise to a Cartan decomposition

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$$

and an involutive automorphism $\sigma = \mathrm{id}|_{\mathcal{H}} \oplus (-\mathrm{id})|_{\mathcal{Q}}$ to a reductive symmetric space decomposition

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{Q}$$

with

$$[\mathcal{H},\mathcal{H}]\subset\mathcal{H},\qquad [\mathcal{H},\mathcal{Q}]\subset\mathcal{Q},\qquad [\mathcal{Q},\mathcal{Q}]\subset\mathcal{H}.$$

We choose them in such a way that $[\sigma,\theta]=0$.

We now consider Iwasawa decompositions $\mathcal{H}=\mathcal{A}_{\mathcal{H}}\oplus\mathcal{N}_{\mathcal{H}}\oplus\mathcal{K}_{\mathcal{H}}$ and $\mathcal{G}=\mathcal{A}\oplus\mathcal{N}\oplus\mathcal{K}$ taken in such a way that $\mathcal{A}_{\mathcal{H}}\subset\mathcal{A}$ and $\mathcal{N}_{\mathcal{H}}\subset\mathcal{N}$. We denote by A, N, K, A_H, N_H and K_H the analytic connected subgroups of G whose Lie algebras are $\mathcal{A}$, $\mathcal{N}$, $\mathcal{K}$, $\mathcal{A}_{\mathcal{H}}$, $\mathcal{N}_{\mathcal{H}}$, and $\mathcal{K}_{\mathcal{H}}$ respectively. We denote by $\overline{N}$ the conjugate by θ of N.

The subgroups $R\equiv AN$ and $\overline{R}\equiv A\overline{N}$ are minimal parabolic (solvable) subgroups of G, as mentioned in Definition 1. The closed orbits of these two groups define the singularity.

3. Causal structure and black hole existence

Definition 3. A *light ray* is a light-like geodesic.

Lemma 4. *Let E be a nilpotent element in $\mathcal{Q}$, and let $\pi: G\to G/H$ be the canonical projection; then a light ray through $[g]\in \mathrm{AdS}_l$ has the form*

$$s(t)=\pi\left(g\mathrm{e}^{-t\,\mathrm{Ad}(k)E}\right) \tag{1}$$

for a certain $k\in K_H$.

Proof. The general form

$$s(t)=\pi\left(\mathrm{e}^{tX}\right)$$

is proven in Theorem 3.2 of chapter XI in [16].

Any vector of the form $\mathrm{Ad}(k)E$ has (Killing) zero norm because the trace of a nilpotent matrix vanishes. Now it is easy to see that the only nilpotent matrices in $\mathcal{Q}$ are of the form q_0+q_i for a certain i. Then $\mathrm{Ad}(k)E=q_0+w_1q_1+\cdots+w_{l-1}q_{l-1}$. From explicit computation given in the Appendix, $\mathrm{Ad}(k)E$ is a generic zero normed vector in $\mathcal{Q}$; see Eq. (A.8). □

Points $s(t)$ with $t\in\mathbb{R}^+$ are said to lie on future-directed light rays issued from g.

3.1. Search for closed R-orbits

Let us start this section by computing the closed orbits of the action of AN and $A\overline{N}$ on AdS_l. In order to see if $x=[g]\in M$ lies in a closed orbit of AN, we "compare" the basis $\{d\pi dL_g q_i\}$ of T_xM and the space spanned by the fundamental vectors of the action. If these two spaces are the same, then x belongs to an open orbit (because a submanifold is open if and only if it has the same dimension as the main manifold). This idea is precisely contained in the following proposition.

Proposition 5. *If R is a subgroup of G with Lie algebra $\mathcal{R}$, then the orbit $R\cdot\vartheta$ is open in G/H if and only if the projection $\mathtt{pr}:\mathcal{R}\to\mathcal{Q}$ is surjective.*

In order to check the openness of the R-orbit of $[g]$, we look at the openness of the $\mathrm{Ad}(g^{-1})R$-orbit of ϑ using the proposition.

A great simplification is possible. The AN-orbits are trivially AN-invariant. So the K part of $[g]=ank$ alone fixes the orbit to which $[g]$ belongs. In the explicit parametrization of K, we know that the $SO(n)$ part is "killed" by the quotient with respect to $SO(1,n)$. Thus we are left with *at most* one AN-orbit for each element in $SO(2)$. Computations using Proposition 5 show that the closed orbits are given by

$$\mathscr{S}=\{\pm[AN],\pm[A\overline{N}]\}. \tag{2}$$

3.2. More about the light cone

Lemma 4 claims that if E is nilpotent in $\mathcal{Q}$, then $\{\mathrm{Ad}(k)E\}_{k\in K_H}$ is the set of all the light-like vectors in $T_{[\vartheta]}\mathrm{AdS}_l \simeq \mathcal{Q}$. So we define the *future light cone* of ϑ by

$$C^+_{[\vartheta]} = \{\pi(\mathrm{e}^{-t\mathrm{Ad}(k)E})\}_{\substack{t\in\mathbb{R}^+\\ k\in K_H}}$$

and that of a general element $[g] \in \mathrm{AdS}_l$ is obtained by the (isometric) action of g thereon:

$$C^+_{\pi(g)} = \{\pi(g\mathrm{e}^{-t\mathrm{Ad}(k)E})\}_{\substack{t\in\mathbb{R}^+\\ k\in K_H}}. \tag{3}$$

The denomination "future" refers to the fact that it only contains positive t. Past light cones correspond to negative t. It should be noted that this definition is independent of the choice of the representative g in the class $\pi(g)$ because, for any $h \in H$, $\pi(gh\mathrm{e}^{-t\mathrm{Ad}(k)E}) = \pi(gh\mathrm{e}^{-t\mathrm{Ad}(k)E}h^{-1})$ which is simply a reparametrization in K_H.

3.3. Computation of the singularity

We here explicitly use the description of AdS_l using the fundamental (defining) representation of $SO(2, n-1)$, i.e. in terms of the embedding coordinates $(u, t, x, y, x_3, \ldots, x_{l-1}) \in \mathbb{R}^{2,l-1}$, and the choices of generators related in the Appendix.

Proposition 6. *In terms of the embedding of* AdS_l *in* $\mathbb{R}^{2,l-1}$*, the closed orbits of* $AN \subset SO(2, l-1)$ *are located at* $y - t = 0$*. Similarly, the closed orbits of* $A\overline{N}$ *correspond to* $y + t = 0$*. In other words, the equation*

$$t^2 - y^2 = 0 \tag{4}$$

describes the singularity $\mathscr{S} = \mathscr{S}_{AN} \cup \mathscr{S}_{A\overline{N}}$.

Proof. The different fundamental vector fields of the AN action can be computed using $X^*_{[g]} = -Xg\cdot\vartheta$. For example, in AdS_3,

$$\begin{aligned} M^*_{[g]} &= \begin{pmatrix} 0 & -1 & 0 & 1\\ 1 & 0 & -1 & 0\\ 0 & -1 & 0 & 1\\ 1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} u\\ t\\ x\\ y \end{pmatrix} = \begin{pmatrix} -t+y\\ u-x\\ -t+y\\ u-x \end{pmatrix}\\ &= (y-t)\partial_u + (u-x)\partial_t + (y-t)\partial_x + (u-x)\partial_y. \end{aligned}$$

The full results are

$$J_1^* = -y\partial_t - t\partial_y \tag{5a}$$
$$J_2^* = -x\partial_u - u\partial_x \tag{5b}$$
$$M^* = (y-t)\partial_u + (u-x)\partial_t + (y-t)\partial_x + (u-x)\partial_y \tag{5c}$$
$$L^* = (y-t)\partial_u + (u+x)\partial_t + (t-y)\partial_x + (u+x)\partial_y \tag{5d}$$
$$W_i^* = -x_i\partial_t - x_i\partial_y + (y-t)\partial_i \tag{5e}$$
$$V_j^* = -x_j\partial_u - x_j\partial_x + (x-u)\partial_j, \quad i, j = 3, \ldots, l-1. \tag{5f}$$

First consider points satisfying $t - y = 0$. It is clear that, at these points, the l vectors J_1^*, M^*, L^* and W_i^* are linearly dependent. Then, there are at most $l-1$ linearly independent vectors amongst the $2(l-1)$ vectors (5); thus the points belong to a closed orbit.

We now show that a point with $t - y \neq 0$ belongs to an open orbit of AN. It is easy to see that J_1^*, L^* and M^* are three linearly independent vectors. The vectors V_i^* give us $l-3$ more. Then they span an l-dimensional space.

The same can be done with the closed orbits of $A\overline{N}$. The result is that a point belongs to a closed orbit of $A\overline{N}$ if and only if $t + y = 0$. $\square$

For the three-dimensional case, it was shown in [4,11] that the non-rotating BTZ black hole singularity is precisely given by Eq. (4). Hence, the following is a particular case of Theorem 2:

Corollary 7. *The non-rotating BTZ black hole is a causal symmetric solvable black hole.*

3.4. Existence of a horizon

We first consider points of the form $K\cdot\vartheta$, which are parametrized by an angle μ. Up to the choice of this parametrization, a light-like geodesic through μ is given by

$$K\cdot e^{-s\mathrm{Ad}(k)E}\cdot\vartheta \tag{6}$$

with $k\in SO(l-1)$ and $s\in\mathbb{R}$.

This geodesic reaches $\mathscr{S}_{AN}$ and $\mathscr{S}_{A\overline{N}}$ for values s_{AN} and $s_{A\overline{N}}$ of the affine parameter, given by

$$s_{AN}=\frac{\sin\mu}{\cos\mu-\cos\alpha},\quad\text{and}\quad s_{A\overline{N}}=\frac{\sin\mu}{\cos\mu+\cos\alpha} \tag{7}$$

where $\cos\alpha$ is the w_2 element of the matrix in (A.8) ($-1\le w_2\le 1$).

Because the part $\sin\mu=0$ is $\mathscr{S}_{AN}$, we may restrict ourselves to the open connected domain of AdS_l given by $\sin\mu>0$. More precisely, $\sin\mu=0$ is the equation of $\mathscr{S}_{AN}$ in the ANK decomposition. In the same way, $\mathscr{S}_{A\overline{N}}$ is given by $\sin\mu'=0$ in the $A\overline{N}K$ decomposition. In order to escape the singularity, the point μ needs $s_{AN},s_{A\overline{N}}<0$. It is only possible to find directions (i.e. an angle α) which respect this condition when $\cos\mu<0$. So the point $\cos\mu=0$ is one point of the horizon.

3.5. A characterization of the horizon

Let $D[g]$ be the set of light-like directions (vectors in $SO(n)$) for which the point $[g]$ falls into $\mathscr{S}_{AN}$. Similarly, $\overline{D}[g]$ is the set of directions for which it falls into $\mathscr{S}_{A\overline{N}}$. It is actually possible to express $\overline{D}$ in terms of D. Indeed

$$\begin{aligned} k\in\overline{D}[g]\quad&\text{iff }\pi(ge^{tk\cdot E})\in\mathscr{S}_{A\overline{N}}\\ &\text{iff}\quad\pi(\theta(g)\theta(e^{tk\cdot E_1}))\in\mathscr{S}_{AN}\\ &\text{iff}\quad\theta(k)\in D(\theta[g])\\ &\text{iff}\quad k\in(D(\theta[g]))_\theta.\end{aligned} \tag{8}$$

So

$$\overline{D}[g]=(D\theta[g])_\theta \tag{9}$$

where the definition of k_θ is

$$\theta(\mathrm{Ad}(k)E)=\mathrm{Ad}(k_\theta)E.$$

This definition is possible because θ is an inner automorphism.

It is easy to see that θ changes the sign of the spatial part of k, i.e. changes $w_i\to -w_i$.

How do we express the condition $g\in\mathscr{H}$ in terms of $D[g]$? The condition for being in the black hole is $D[g]\cup\overline{D}[g]=SO(n)$. If the complementary of $D[g]\cup\overline{D}[g]$ has an interior (i.e. if it contains an open subset), then by continuity the complementary $D[g']\cup\overline{D}[g']$ has also an interior for all $[g']$ near $[g]$. In this case, $[g]$ cannot belong to the horizon. So a characterization of $\mathscr{H}$ is the fact that the boundary of $D[g]$ and $\overline{D}[g]$ coincide. Eq. (9) shows that $\mathscr{H}$ is θ-invariant.

We can explicitly express $D[\mu]$ for $\mu\in SO(2)$ by examining Eq. (7). Let us write w_2 instead of $\cos\alpha$. The set $D[\mu]$ is the set of $w_2\in[-1,1]$ such that $\cos\mu-w_2>0$:

$$D[\mu]=[-1,\cos\mu[. \tag{10}$$

So in order for $[\mu]$ to belong to $\mathscr{H}$, it must satisfy

$$D[\theta]_\theta=[-1,\cos\mu'[_\theta=]-\cos\mu',1].$$

Consequently, if μ is the K component of g in the ANK decomposition and μ' that of θu, then we can describe the horizon by

$$\cos\mu = -\cos\mu'. \tag{11}$$

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Appendix. Explicit matrix choices

The first choice is to parametrize $SO(2,n)$ and $SO(1,n)$ in such a way that the latter leaves unchanged the vector $(1,0,0,\ldots)$. Then

$$\mathcal{H} = \mathfrak{so}(1,n) \rightsquigarrow \begin{pmatrix} \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} \cdots 0 \cdots \\ \leftarrow v^t \rightarrow \end{pmatrix} \\ \begin{pmatrix} \vdots & \uparrow \\ 0 & v \\ \vdots & \downarrow \end{pmatrix} & B \end{pmatrix} \tag{A.1}$$

where v is $n \times 1$ and B is skew symmetric $n \times n$. When we speak about $\mathfrak{so}(n)$, we usually refer to the B part of $\mathcal{H}$. A complementary space $\mathcal{Q}$ such that $[\mathcal{H}, \mathcal{Q}] \subset \mathcal{Q}$ is given by

$$\mathcal{Q} \rightsquigarrow \begin{pmatrix} \begin{matrix} 0 & a \\ -a & 0 \end{matrix} & \begin{pmatrix} \leftarrow w^t \rightarrow \\ \cdots 0 \cdots \end{pmatrix} \\ \begin{pmatrix} \uparrow & \vdots \\ w & 0 \\ \downarrow & \vdots \end{pmatrix} & 0 \end{pmatrix}. \tag{A.2}$$

We consider the involutive automorphism $\sigma = \mathrm{id}_{\mathcal{H}} \oplus (-\mathrm{id})_{\mathcal{Q}}$ and the corresponding symmetric space structure on $\mathcal{G}$. As a basis of $\mathcal{Q}$, we choose q_0 as the 2×2 antisymmetric upper left square and q_i as those obtained with w full of zeros apart from a 1 as the ith component. Next we choice the Cartan involution $\theta(X) = -X^t$ which gives rise to a Cartan decomposition

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{P}.$$

The latter choice is made in such a way that $[\sigma, \theta] = 0$. It can be computed, but it is not astonishing that the compact part $\mathcal{K}$ is made up of "true" rotations while $\mathcal{P}$ contains the boost. So

$$\mathcal{K} = \begin{pmatrix} \mathfrak{so}(2) & \\ & \mathfrak{so}(n) \end{pmatrix},$$

where elements of $SO(2)$ are represented as

$$\begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix}.$$

A common abuse of notation in the text is to identify the angle μ with the element of $SO(2)$ itself.

In order to build an Iwasawa decomposition, one has to choose a maximal abelian subalgebra $\mathcal{A}$ of $\mathcal{P}$. Since rotations are in $\mathcal{K}$, they must be boosts, and the fact that there are only two time-like directions restricts $\mathcal{A}$ to a two-dimensional algebra. Up to reparametrization, it is thus generated by $u\partial_x + x\partial_u$ and $t\partial_y + y\partial_t$. Our matrix choices are

$$J_1 = \begin{pmatrix} & 0 & & \\ 0 & 0 & 0 & 1 \\ & 0 & & \\ & 1 & & \end{pmatrix} \in \mathcal{H}, \quad \text{and} \quad J_2 = q_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & & & \\ 1 & & & \\ 0 & & & \end{pmatrix} \in \mathcal{Q}.$$

From here, we have to build root spaces. There still remain a lot of arbitrary choices — among them, the positivity notion on the dual space $\mathcal{A}^*$. An element X in $\mathcal{G}_{(a,b)}$ fulfills $\mathrm{ad}(X)J_1 = aJ_1$ and $\mathrm{ad}(X)J_2 = bJ_2$. The symbol E_{ij} denotes the matrix full of zeros with a 1 as the component ij. The results are

$$\mathcal{G}_{(0,0)} \rightsquigarrow \begin{pmatrix} & & x & 0 & \\ & & 0 & y & \\ x & 0 & & & \\ 0 & y & & & \\ & & & & D \end{pmatrix}, \tag{A.3}$$

where $D \in M_{(n-2)\times(n-2)}$ is skew symmetric,

$$\mathcal{G}_{(1,0)} \rightsquigarrow W_i = E_{2i} + E_{4i} + E_{i2} - E_{i4}, \tag{A.4a}$$

$$\mathcal{G}_{(-1,0)} \rightsquigarrow Y_i = -E_{2i} + E_{4i} - E_{i2} - E_{i4}, \tag{A.4b}$$

$$\mathcal{G}_{(0,1)} \rightsquigarrow V_i = E_{1i} + E_{3i} + E_{i1} - E_{i3}, \tag{A.4c}$$

$$\mathcal{G}_{(0,-1)} \rightsquigarrow X_i = -E_{1i} + E_{3i} - E_{i1} - E_{i3} \tag{A.4d}$$

with $i : 5 \to n+2$ and

$$\mathcal{G}_{(1,1)} \rightsquigarrow M = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix}, \qquad \mathcal{G}_{(1,-1)} \rightsquigarrow L = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}, \tag{A.5}$$

$$\mathcal{G}_{(-1,1)} \rightsquigarrow N = \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}, \qquad \mathcal{G}_{(-1,-1)} \rightsquigarrow F = \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{pmatrix}. \tag{A.6}$$

The choice of positivity is

$$\mathcal{N} = \{V_i, W_j, M, L\}. \tag{A.7}$$

The following result is important in the computation of the light cones: if $k \in SO(n)$, then the choice $E = q_0 + q_2$ of the nilpotent element in $\mathcal{Q}$ gives

$$\mathrm{Ad}(k)E = \begin{pmatrix} 0 & 1 & w_1 & w_2 & \dots \\ -1 & & & & \\ w_1 & & & & \\ w_2 & & & & \\ \vdots & & & & \end{pmatrix} \tag{A.8}$$

where the elements w_i are restricted by the condition $\sum_{i=1}^{l-1} w_i^2 = 1$.

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